

# Gap theorems for Kähler-Ricci solitons

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**Abstract** In this paper, we prove that a gradient shrinking compact Kähler-Ricci soliton cannot have too large Ricci curvature unless it is Kähler-Einstein.

## 1 Introduction

The main purpose of this paper is to give two gap theorems on gradient shrinking Kähler-Ricci solitons with positive Ricci curvature on a compact Kähler manifold with  $c_1(M) > 0$ . Here we only discuss compact manifolds.

Since Ricci flow was introduced by R. Hamilton in [4], Ricci solitons have been studied extensively. One interesting question is to classify all the Ricci solitons with some curvature conditions, especially with positive curvature operator. It is well known that there is no expanding or steady Ricci solitons on a compact Riemannian manifold of any dimension, and no shrinking Ricci solitons of dimension 2 and 3 (cf. [5][6]). Recently Böhm-Wilking in [1] extended Hamilton's maximum principles, and they essentially proved that there is no nontrivial shrinking Ricci solitons with positive curvature operator on a compact manifold.

For the Kähler case, by Siu-Yau's result in [10] a Kähler-Ricci soliton with positive holomorphic bisectional curvature is Kähler-Einstein. It is still very interesting to know how to prove this result via Kähler-Ricci flow (cf. [14]), or via complex Monge-Ampère equations. Rotationally symmetric Kähler-Ricci solitons have been constructed by Koiso [7] and Cao [2]. The existence and uniqueness of Kähler-Ricci solitons have been extensively studied in literature (cf. [3][12][13][15]).

Let  $M$  be a compact Kähler manifold with  $c_1(M) > 0$ . A Kähler metric  $g$  with its Kähler form  $\omega$  is called a Kähler-Ricci soliton with respect to a holomorphic vector field  $X$  if the equation

$$\text{Ric}(\omega) - \omega = L_X \omega$$

is satisfied. Since  $\omega$  is closed, we may write

$$L_X \omega = -\sqrt{-1} \partial \bar{\partial} u$$

for some function  $u$  with  $u_{ij} = u_{\bar{i}\bar{j}} = 0$ . Then the Futaki invariant is given by

$$f_X = -\frac{1}{V} \int_M X u \omega^n = \frac{1}{V} \int_M |\nabla u|^2 \omega^n > 0.$$

Note that  $f_X$  depends only on the Kähler class  $[\omega]$  and the holomorphic vector field  $X$ .

The following is the first result in this paper:

**Theorem 1.1.** *Let  $\omega$  be a Kähler-Ricci soliton with a holomorphic vector field  $X$ . If*

$$|Ric(\omega) - \omega| < \frac{-f_X + \sqrt{f_X^2 + 4f_X}}{2}, \quad (1.1)$$

*then  $\omega$  is Kähler-Einstein.*

*Remark 1.2.* Theorem 1.1 tells us that the Ricci curvature of a Kähler-Ricci soliton can not be too close to the Kähler form, so it gives a gap between Kähler-Ricci solitons and Kähler-Einstein metrics.

In [11], G. Tian proposed a conjecture on the solution  $\omega_t$  of complex Monge-Ampère equations on a Kähler manifold with  $c_1(M) > 0$

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\omega - t\varphi}\omega^n, \quad \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \quad (1.2)$$

where  $h_\omega$  is the Ricci potential with respect to the metric  $\omega$ . It is known that  $M$  admits a Kähler-Einstein metric if and only if (1.2) is solvable for  $t \in [0, 1]$ . On the other hand, if  $M$  admits no Kähler-Einstein metrics, then (1.2) is solvable for  $t \in [0, t_0)$  ( $t_0 \leq 1$ ). Tian conjectured that when this case occurs,  $(M, \omega_t)$  converges to a space  $(M_\infty, \omega_\infty)$ , which might be a Kähler-Ricci soliton after certain normalization. Observe that the metric  $\omega_t$  of (1.2) has  $Ric(\omega_t) > (1 - t)\omega_t$ , the following theorem tells us that if Tian's conjecture is true,  $t_0$  might not be close to 1.

**Theorem 1.3.** *Let  $\omega$  be a Kähler-Ricci soliton with a holomorphic vector field  $X$ . There exists a constant  $\epsilon > 0$  depending only on the Futaki invariant  $f_X$  such that if*

$$Ric(\omega) > (1 - \epsilon)\omega,$$

*then  $\omega$  is Kähler-Einstein.*

*Remark 1.4.* In Theorem 1.3,  $\epsilon$  can be expressed explicitly from the proof.

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## 2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let  $g$  be a Kähler-Ricci soliton with

$$Ric - \omega = -\sqrt{-1}\partial\bar{\partial}u.$$

By the definition of the Futaki invariant,

$$f_X = -\frac{1}{V} \int_M X(u)\omega^n = \frac{1}{V} \int_M |\nabla u|^2 \omega^n > 0.$$

Define

$$\epsilon := \max_M |Ric - \omega|, \quad (2.1)$$

we have the following lemma:

**Lemma 2.1.** *Let  $\lambda_1$  be the first eigenvalue of  $\Delta_g$ , then  $\lambda_1 \geq 1 - \epsilon$ .*

*Proof.* Let  $f$  be an eigenfunction satisfying  $\Delta_g f = -\lambda_1 f$ , then

$$0 \leq \int_M |\nabla \nabla f|^2 \omega^n = \int_M \left( (\Delta_g f)^2 - Ric(\nabla f, \bar{\nabla} f) \right) \omega^n \leq (\lambda_1^2 - (1 - \epsilon)\lambda_1) \int_M f^2 \omega^n.$$

Thus,  $\lambda_1 \geq 1 - \epsilon$  and the lemma is proved.  $\square$

By Lemma 2.1, we have

$$\frac{1}{V} \int_M |\nabla \bar{\nabla} u|^2 \omega^n \geq (1 - \epsilon) \frac{1}{V} \int_M |\nabla u|^2 \omega^n = (1 - \epsilon) f_X. \quad (2.2)$$

In fact, we can write  $u = \sum_{i=1}^{\infty} c_i f_i$ , where  $f_i$  are the eigenfunctions of  $\Delta_g$  such that

$$\Delta_g f_i = -\lambda_i f_i, \quad \int_M f_i^2 \omega^n = V.$$

Here  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m < \dots$ . Then

$$\Delta_g u = - \sum_{i=1}^{\infty} c_i \lambda_i f_i.$$

Integrating by parts, we have

$$\frac{1}{V} \int_M |\nabla \bar{\nabla} u|^2 \omega^n = \frac{1}{V} \int_M (\Delta_g u)^2 \omega^n = \sum_{i=1}^{\infty} c_i^2 \lambda_i^2 \geq \lambda_1 \sum_{i=1}^{\infty} c_i^2 \lambda_i = \frac{1}{V} \int_M |\nabla u|^2 \omega^n.$$

Hence the inequality (2.2) holds. On the other hand, by (2.1) we have

$$\frac{1}{V} \int_M |\nabla \bar{\nabla} u|^2 \omega^n \leq \epsilon^2.$$

Combining this together with (2.2), we have

$$\epsilon^2 - (1 - \epsilon) f_X \geq 0.$$

Then

$$\epsilon \geq \frac{-f_X + \sqrt{f_X^2 + 4f_X}}{2}.$$

The theorem is proved.

### 3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Let  $g$  be a Kähler Ricci soliton with

$$Ric(\omega) - \omega = -\sqrt{-1}\partial\bar{\partial}u,$$

where  $u_{ij} = u_{\bar{i}\bar{j}} = 0$ . We can normalize  $u$  such that

$$\int_M u \omega^n = 0. \quad (3.1)$$

**Lemma 3.1.**  *$u$  satisfies the following equation*

$$\Delta_g u + u - |\nabla u|^2 = -f_X. \quad (3.2)$$

*Proof.* By direct calculation, we have

$$(u_{i\bar{i}} + u - u_i u_{\bar{i}})_j = u_{i\bar{i}j} + u_j - u_i u_{\bar{i}j} = -R_{j\bar{k}} u_k + u_j - u_i u_{\bar{i}j} = 0.$$

Then  $\Delta_g u + u - |\nabla u|^2$  is a constant. By (3.1) and the definition of the Futaki invariant, (3.2) holds and the lemma is proved.  $\square$

Now we assume  $Ric(\omega) \geq \lambda \omega$  with  $\lambda > 0$ , so the scalar curvature  $R \geq n\lambda$ . By Lemma 3.1,  $u$  is bounded from below. In fact,

$$u = -f_X + |\nabla u|^2 - \Delta u = -f_X + |\nabla u|^2 + R - n \geq -f_X + n\lambda - n. \quad (3.3)$$

Now we can prove the following lemma

**Lemma 3.2.** *The scalar curvature is uniformly bounded from above, i.e.*

$$R \leq \Lambda(\lambda, f_X),$$

where  $\Lambda(\lambda, f_X)$  is a constant depending only on  $\lambda$  and  $f_X$ . Moreover,  $\lim_{\lambda \rightarrow 1} \Lambda(\lambda, f_X)$  is finite.

*Proof.* By the Kähler-Ricci soliton equation (3.2), we have

$$|\nabla u|^2 = \Delta_g u + u + f_X = n - R + u + f_X \leq u + n + f_X - n\lambda. \quad (3.4)$$

Let  $B = f_X - n\lambda + n + 1$ , then by (3.3)  $u + B \geq 1$ . Hence

$$\frac{|\nabla u|^2}{u + B} \leq 1 - \frac{1}{u + B} \leq 1.$$

Let  $p \in M$  be a minimum point of  $u$ , by the normalization condition (3.1)  $u(p) \leq 0$ . Then for any  $x \in M$ , we have

$$\sqrt{u + B}(x) - \sqrt{u + B}(p) \leq |\nabla \sqrt{u + B}| \operatorname{diam}(g) \leq \frac{1}{2} \frac{|\nabla u|}{\sqrt{u + B}} \frac{c_n \pi}{\sqrt{\lambda}} \leq \frac{c_n \pi}{2\sqrt{\lambda}}, \quad (3.5)$$

where  $\text{diam}(g)$  is the diameter of  $(M, g)$  and  $c_n$  is a constant depending only on  $n$ . It follows that

$$u \leq \frac{c_n^2 \pi^2}{4\lambda} + c_n \pi \sqrt{\frac{B}{\lambda}}.$$

Therefore,

$$R = n - \Delta u = n + f_X - |\nabla u|^2 + u \leq n + f_X + \frac{c_n^2 \pi^2}{4\lambda} + c_n \pi \sqrt{\frac{B}{\lambda}}.$$

□

Now we can finish the proof of Theorem 1.3. By Lemma 3.2 and  $R \geq n\lambda$ , we have

$$\begin{aligned} \frac{1}{V} \int_M |\nabla \bar{\nabla} u|^2 \omega^n &= \frac{1}{V} \int_M (R - n)^2 \omega^n \\ &= \frac{1}{V} \int_{\{R > n\}} (R - n)^2 \omega^n + \frac{1}{V} \int_{\{R < n\}} (R - n)^2 \omega^n \\ &\leq (\Lambda - n) \frac{1}{V} \int_{\{R > n\}} (R - n) \omega^n + n^2 (1 - \lambda)^2. \end{aligned}$$

On the other hand,

$$0 = \int_{\{R > n\}} (R - n) \omega^n + \int_{\{R < n\}} (R - n) \omega^n.$$

Therefore,

$$\begin{aligned} \frac{1}{V} \int_M |\nabla \bar{\nabla} u|^2 \omega^n &= (\Lambda - n) \frac{1}{V} \int_{\{R < n\}} (n - R) \omega^n + n^2 (1 - \lambda)^2 \\ &\leq (\Lambda - n) n (1 - \lambda) + n^2 (1 - \lambda)^2 \\ &\rightarrow 0, \end{aligned} \tag{3.6}$$

as  $\lambda \rightarrow 1$ . On the other hand, by the inequality (2.2) we have

$$\frac{1}{V} \int_M |\nabla \bar{\nabla} u|^2 \omega^n \geq \lambda f_X > 0,$$

which contradicts (3.6) when  $\lambda$  is sufficiently close to 1. The theorem is proved.

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